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CONVERGENCE AND ASYMPTOTIC AGREEMENT IN DISTRIBUTED DECISION PR--ETC(U)
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# CONVERGENCE AND ASYMPTOTIC AGREEMENT IN DISTRIBUTED DECISION PROBLEMS*

by

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ABSTRACT

We consider a distributed team decision problem in which different agents obtain from the environment different stochastic measurements, possibly at different random times, related to the same uncertain random vector. Each agent has the same objective function and prior probability distribution. We assume that each agent can compute an optimal tentative decision based upon his own observation and that these tentative decisions are communicated and received, possibly at random times, by a subset of other agents. Conditions for asymptotic convergence of each agent's decision sequence and asymptotic agreement of all agents' decisions are derived.

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* Research was supported by the Office of Naval Research under grant ONR/N00014-77-C-0532 (NR 041-519)

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This paper was submitted to the 1982 IEEE Conference on Decision and Control and to the IEEE Transactions on Automatic Control.

## 1. INTRODUCTION.

Consider the following situation: A set $\{1, \ldots, N\}$ of $N$ agents possessing a common model of the world (same prior probabilities) and having the same cost function want to make a decision. Each agent bases his decision on a set of observations he has obtained and we allow these observations to be different for each agent. Given this setting, the decisions of the agents will be generally unrelated. Aumann [1] has shown, however, that agreement is guaranteed in the following particular case: If the decision to be made is the evaluation of the posterior probability of some event and if all agents' posteriors are common knowledge, then all agents agree. (In Aumann's terminology, common knowledge of an event means that all agents know it, all agents know that all agents know it, and so on, ad infinitum. )

The situation where each agent's posterior is common knowledge is very unlikely, in general. On the other hand, if agreement is to be guaranteed, posteriors have to be common knowledge. The problem then becomes how to reach a state of agreement where decisions are common knowledge, starting from an initial state of disagreement.

Geanakoplos and Polemarchakis [6] and Borkar and Varaiya [5] gave the following natural solution to the above problem: Namely, agents start communicating to each other their tentative posteriors (or, in the formulation of [5] the conditional expectation of a fixed random variable) and then update their own posterior, taking into account the new information they have received. In the limit, each person's posterior converges (by the martingale convergence theorem) and assuming that "enough" communications have taken place, they all have to converge to a common limit.

The above results hold even when each agent obtains additional raw observations during the adjustment process and when the history of communications is itself random. Similar results were also proved for a detection problem [5].

A related – and much more general situation – is the subject of this paper; we assume that the agents are not just interested in obtaining an optimal estimate or a likelihood ratio, but their objective is to actually minimize an arbitrary cost function. In this setting, it is reasonable to assume that agents communicate to each other tentative decisions. That is, at any time, an agent computes an optimal decision given the information he possesses and communicates it to other agents. Whenever an agent receives such a message from another agent, his information essentially increases and he will, in general, update his own tentative decision, and so on. In the sequel we prove that the qualitative results obtained in [5], [6] for the estimation problem (convergence and asymptotic agreement) are also valid for the decision making problem for several, quite general, choices of the structure of the

cost function. However, tentative decisions do not form a martingale sequence and a substantially different mathematical approach is required. We point out that estimation problems are a special case of the decision problems studied in this paper, being equivalent to the minimization of the mean square error.

A drawback of the above setting is that each agent is assumed to have an infinite memory. We have implicitly assumed that the knowledge of an agent can only increase with time and, therefore, he has to remember the entire sequence of messages he has received in the past. There is also the implicit assumption that if an agent receives additional raw data from the environment, while the communication process is going on, these data are remembered forever. These assumptions are undesirable, especially if the agents are supposed to be humans, because limited memory is a fundamental component of the bounded rationality behavior of human decision makers. We will therefore relax the infinite memory assumption and allow the agents to forget any portion of their past knowledge. We only constrain them to remember their most recent decision and the most recent message (tentative decision) coming from another agent. We then obtain results similar to those obtained for the unbounded memory model, although in a slightly weaker sense.

A particular problem of interest is one in which all random variables are jointly Gaussian and the cost is a quadratic function of an unknown state of the world and the decision. It was demonstrated in [5] that the common limit to which decisions converge (for the estimation problem) is actually the centralized estimate, i.e. the estimate that would be obtained if all agents were to communicate their detailed observations. We prove (section 4) that the same is true in the presence of memory limitations, provided that each agent never forgets his own raw observations. (That is, he may only forget past tentative decisions sent to him by other agents.)

We end this section with a discussion of the nature of our model and the possible range of its validity. The first point to be made is that there are strong underlying rationality assumptions if we postulate that our model describes the way that humans update their decisions, even if we constrain them to have limited memory. The decision of an agent at some stage is the optimal decision given his observations and some tentative decisions of other agents that have been communicated to him. Except for particular problems such as linear quadratic gaussian decision problems, this corresponds to a very hard optimization problem. The reason is that the other agents' tentative decisions represent an indirect way of communicating information. Essentially, when agent $A$ receives agent $B$'s decision, he tries to deduce $B$'s observations from $B$'s decision. So, one agent combines information to obtain an optimal decision and the other agent tries to invert this map and recover the original information. While this is mathematically meaningful, it is, in general, beyond the information

processing capabilities of human decision makers. In view of the above, we propose the following interpretation: while it is not true that human decision makers will actually do the computations prescribed by the model, a group of well-trained individuals, very familiar with the decision problem at hand will make tentative decisions close to the ones predicted by the model and will agree in the limit. Bacharach [2, p.183] gives some support to the above statement by citing evidence that "in practice awareness of others' opinions tends to modify one's own: in the case of factual questions a general tendency to converge has often been observed".

A second weak point of the model is that not only each agent has the same prior information and knows the statistics of the other agents' observations but also has the same model of the probabilistic mechanism that generates inter-agent communications. In particular, if this is a deterministic mechanism, an agent must know the precise history of communications between any pair of other agents, a very strong requirement. This weakness disappears, however, if every tentative decision is broadcasted simultaneously to all other agents. This will be the case, for example, if a set of experts with the same objective teleconfer and take turns into suggesting what they believe to be the optimal decision.

Finally, we point out that our scheme is fundamentally different from schemes for distributed decision making and computation in which each agent controls his own decision variable and the cost function depends jointly on all agents' control variables [3, 7]. What we have here is a common decision variable which has to be jointly fixed by a set of decision makers.

## II. MODEL FORMULATION.

In this section we present a mathematical formulation of the model informally described in the introduction. We start with a most general set of assumptions and later proceed to the development of alternative specialized models to be considered (e.g. memory limitations, particular forms of the cost function etc.). As far as the description of the sequence of communications and updates goes, we basically adopt the model of Borkar and Varaiya [5] except that time is considered to be discrete. As in [5], events are timed with respect to a common, absolute clock. As far as notation is concerned, we will use subscripts to denote time and superscripts to denote agents.

We assume that we are given a set $\{1,\ldots,N\}$ of $N$ agents, an underlying probability space $(\Omega,\mathcal{F},\mathcal{P})$ and a scalar valued cost function $c:\Omega\times U\mapsto R$, where $U$ is the set of admissible values of the decision variable. It will be useful in the sequel to distinguish between elements of $U$ and $U$-valued random variables. The letter $v$ will be used to denote elements of $U$ whereas $u$, $w$ will be used to denote $U$-valued random variables (measurable functions from $\Omega$ to $U$).

Our first assumption is of a purely technical nature and is satisfied by almost all probability spaces arising in applications. The following two assumptions, however, are of a more essential nature.

Assumption 1: $\Omega$ is a Polish (complete, separable, metric) space and $\mathcal{F}$ is the associated Borel $\sigma$-field (the $\sigma$-field generated by the open subsets of $\Omega$).

Assumption 2: Either

(2.1): $U$ is a finite set, or

(2.2): $U$ is a closed, convex subset of $R^n$.

Assumption 3: The cost function $c$ is nonnegative and jointly measurable in $(\omega,v)$. Moreover, $E[c(v)]<\infty$, $\forall v\in U$. When assumption (2.2) holds, we assume that there exists a positive, measurable and integrable function $A:\Omega\mapsto R$ such that

$$A(\omega)||v_1-v_2||^2\leq\frac{1}{2}[c(\omega,v_1)+c(\omega,v_2)]-c\left(\omega,\frac{v_1+v_2}{2}\right),\qquad \forall\omega\in\Omega,\quad \forall v_1,v_2\in U. \tag{1}$$

The last part of assumption 3 states that $c$ is a strictly convex function of $v$ and strict convexity holds in a uniform way, for any fixed $\omega\in\Omega$. This assumption is satisfied, in particular, if $c$ is twice continuously differentiable in $v$ and its Hessian is positive definite, uniformly in $v$, for any fixed $\omega\in\Omega$. It is also satisfied if $c$ is strictly convex and $U$ is a compact set.

We may use the function $A$, defined in assumption 3, to define a new measure $\mu$ on $(\Omega,\mathcal{F})$ by

$$\mu(B)=\int_B A(\omega)\,d\mathcal{P}(\omega),\qquad B\in\mathcal{F}. \tag{2}$$

This measure will be very useful in section 3.

We now consider the generic situation facing agent $i$ at some time $n$. Let $\mathcal{F}_n^i \subset \mathcal{F}$ be a $\sigma$-field of events describing the information possessed by agent $i$ at time $n$. Because of assumption 3, the conditional expectation $E[c(v)|\mathcal{F}_n^i]$ exists (is finite), is $\mathcal{F}_n^i$-measurable and is uniquely determined up to a set of measure zero [10], for any fixed $v \in U$. Agent $i$ then computes a <u>tentative decision</u> $u_n^i$ that minimizes $E[c(v)|\mathcal{F}_n^i]$.

We are now forced into a mathematical digression that will ensure that $u_n^i$ is well-defined and determines a $\mathcal{F}_n^i$-measurable random variable.

We need to pay special attention only for the convex, continuous problem (assumptions 2.2 and 3). We introduce another technical assumption describing the kind of knowledge an agent may possess at some point in time. Similarly with assumption 1, it poses no real restriction. Roughly speaking, it states that the knowledge of an agent can be described by a Polish probability space.

<u>Assumption 4:</u> If $\mathcal{F}_n^i \subset \mathcal{F}$ describes the knowledge of agent $i$ at time $n$, there exists a Polish probability space $(\Omega^*, \mathcal{F}^*, \mathcal{P}^*)$ and a mapping $f:\Omega \mapsto \Omega^*$ such that $\mathcal{F}_n^i$ is the smallest $\sigma$-field that makes $f$ measurable and $\mathcal{P}^*(B) = \mathcal{P}(f^{-1}(B))$ , $\forall B \in \mathcal{F}^*$.

We then have the following Lemma (proved in the Appendix):

<u>Lemma 1:</u> Under assumptions 1, 2.2, 3, 4, there exists an almost everywhere unique, $\mathcal{F}_n^i$-measurable random variable $u_n^i$ such that

$$E[c(u_n^i)|\mathcal{F}_n^i] \leq E[c(w)|\mathcal{F}_n^i], \qquad \text{almost surely,} \tag{3}$$

for any $U$-valued, $\mathcal{F}_n^i$-measurable random variable $w$. Moreover, $E[c(u_n^i)] = E[c(w)]$ for some $\mathcal{F}_n^i$-measurable $w$ if and only if $u_n^i = w$, almost everywhere. The same results are true, (except for uniqueness) under assumption 2.1.

We continue with a description of the process of communications between agents. When, at time $n$, agent $i$ computes his tentative optimal decision $u_n^i$, he may communicate it to any other agent. (If $u_n^i$ is not unique, a particular minimizing $u_n^i$ is selected according to some commonly known rule.) Whether, when and to which agents $u_n^i$ is to be sent is a random event whose statistics are described by $(\Omega, \mathcal{F}, \mathcal{P})$. We also allow the transmission time to be random but finite. We assume that when an agent receives a message he knows the identity of the agent who sent it.

Note that the time at which an agent receives a message is a random variable on $(\Omega, \mathcal{F}, \mathcal{P})$. As pointed out in [5], this allows the possibility of signalling additional information, beyond that contained in $u_n^i$, by appropriately choosing when and to which agents to communicate.

We now impose conditions on the number of messages to be communicated in the long run; these conditions are necessary for agreement to be guaranteed. Namely, we require that there is an indirect communication link from any agent to any other agent which is used an infinite number of times. This can be made precise as follows:

Let $A(i)$ be the set of all agents that send an infinite number of messages to agent $i$, with probability 1. Then, we make the following assumption:

Assumption 5: There is a sequence $m_1, \ldots, m_{k+1} = m_1$ of not necessarily distinct agents such that $m_i \in A(m_{i+1})$, $i = 1, 2, \ldots, k$. Each agent appears at least once in this sequence.

The main consequence of assumption 5, which will be repeatedly used, is the following: If $\{h^i: 1, \ldots, N\}$ is a set of numbers such that $h^i \leq h^j$, $\forall j \in A(i)$, $\forall i$, then $h^i = h^j$, $\forall i, j$.

We continue with a more detailed specification of the operation of the agents. We introduce assumptions on the knowledge $\mathcal{F}_n^i$ which are directly related to the properties of the memory of agent $i$. An agent may receive (at any time) observations on the state of the world or receive tentative decisions (messages) of other agents. The knowledge of an agent at some time will be a subset (depending on the properties of his memory) of the total information he has received up to that time. We consider three alternative models of memory, formalized with the three assumptions that follow.

Let $w_n^i$ be the last message received by agent $i$ up to (and including) time $n$. Our most general assumption requires that $w_n^i$ and $u_{n-1}^i$ are remembered at time $n$:

Assumption 6: (Imperfect Memory) The $\sigma$-field $\mathcal{F}_n^i$ is such that $u_{n-1}^i$ and $w_n^i$ are $\mathcal{F}_n^i$-measurable.

Remark: Assumption 6 makes sense only because of Lemma 1 which guarantees that $u_{n-1}^i$ and $w_n^i$ are $\mathcal{F}$-measurable. We can then define, inductively, $\mathcal{F}_n^i$ to be the smallest sub-$\sigma$-field of $\mathcal{F}$ with respect to which $u_{n-1}^i$, $w_n^i$ and any other information rememberd are measurable.

Assumption 7: (Imperfect Memory; Own Data Remembered) Let $G_n^i$ be the subfield of $\mathcal{F}$ describing all information that has been observed by agent $i$ up to time $n$, except for the messages of other agents. We assume that assumption 6 holds and that $G_n^i \subset \mathcal{F}_n^i$.

Assumption 8: (Perfect Memory) We let assumptions 6 and 7 hold and assume that $\mathcal{F}_n^i \subset \mathcal{F}_{n+1}^i$, $\forall i, n$.

So, with assumption 6 we only assume that an agent remembers the content of the last message he received, as well as his last tentative decision. With assumption 7 we also assume that he remembers all his past observations but is still allowed to forget the contents of the past messages he received. In this case the total information available to all agents is preserved. Finally, assumption 8 implies that the knowledge of an agent can only increase. Whenever assumption 8 holds, we will denote by

$\mathcal{F}^i_\infty$ the smallest $\sigma$-field in $\mathcal{F}$ containing $\mathcal{F}^i_n$, for all $n$.

We conclude this section by defining a few problems of particular interest:

(i) Estimation Problem: Given a $R^n$–valued random vector $x$ on $(\Omega, \mathcal{F}, \mathcal{P})$, let $U = R^n$ and $c(v) = (x - v)^T(x - v)$, where $T$ denotes transpose. It is easy to see that this is a particular case of a strictly convex function covered by assumption 3, with $A(\omega)$ being a constant.

(ii) Static Linear Quadratic Gaussian Decision Problem (LQG): Let $x$ be an unknown random vector. Let the sequence of transmission and reception times be *deterministic*. We assume that the random variables observed by the agents are, together with $x$, jointly normally distributed. We allow the total number of observations to be infinite, although we consider separately the finite–dimensional problem. Let $U = R^n$ and let the cost function be $c(v) = v^T R v + x^T Q v$, with $R > 0$. It follows that the optimal tentative decision of agent $i$ at time $n$ is $u^i_n = GE[x|\mathcal{F}^i_n] = E[Gx|\mathcal{F}^i_n]$, where $G$ is a precomputable matrix. If we redefine the unknown vector $x$ to be equal to $Gx$ instead of $x$, we conclude that we may restrict to estimation problems, without loss of generality.

(iii) Finite Probability Spaces: Here we let $\Omega$ be a finite set. Then, there exist finitely many $\sigma$-fields of subsets of $\Omega$. It follows that tentative decisions can take finitely many values. We therefore assume, without loss of generality, that $U$ is also a finite set.

## III. CONVERGENCE AND AGREEMENT RESULTS.

In this section we state and discuss our main results. All proofs can be found in the Appendix. Assumptions 1, 3, 4 and 5 will be assumed throughout the rest of the paper and will not be explicitly mentioned in the statement of each Theorem. We start with the least restrictive assumptions on memory:

Theorem 1: Under assumptions 2.2 (convex costs), 6 (imperfect memory) and deterministic transmissions and receptions:
a) $\lim_{n\to\infty}(u^i_{n+1} - u^i_n) = 0$, in probability and in $L_2(\Omega, \mathcal{F}, \mu)$.
b) $\lim_{n\to\infty}(u^i_n - u^j_n) = 0$, $\forall i, j$, in probability and in $L_2(\Omega, \mathcal{F}, \mu)$.

Consider the following situation: At time zero, before any observations are obtained, the sequence of transmissions and receptions is selected in random, according to a statistical law which is independent from all observations to be obtained in the future and from $c(v)$, for any $v \in U$. In other words, communications do not carry any information other than the content of the message being communicated (no signalling allowed). Suppose that the sequence of communications that has been selected becomes known to all agents. From that point on, the situation is identical with that of deterministic communications. In fact, a moment's thought will show that it is sufficient for the history of communications to become commonly known as it occurs: agent $i$ only needs to know, at time $n$, what communications have occured up to that time, so that he can interpret correctly the meaning of the messages he is receiving.

We can formalize these ideas as follows: We are given a product probability space $(\Omega \times \Omega^*, \mathcal{F} \times \mathcal{F}^*, \mathcal{P} \times \mathcal{P}^*)$ where $(\Omega, \mathcal{F}, \mathcal{P})$ satisfies assumptions 1 and 4 (for any sequence of communications) and where $(\Omega^*, \mathcal{F}^*, \mathcal{P}^*)$ describes the communications process. For each $\omega^* \in \Omega^*$, we obtain a distributed decision problem on $(\Omega, \mathcal{F}, \mathcal{P})$ with deterministic communications. In that case:

Theorem 2: Under assumptions 2.2, 6 and independent communications (as described above), $\lim_{n\to\infty}(u^i_{n+1} - u^i_n) = \lim_{n\to\infty}(u^i_n - u^j_n) = 0$, in probability with respect to $\mathcal{P} \times \mathcal{P}^*$.

Strictly speaking, Theorems 1 and 2 do not guarantee convergence of the decisions of each agent. Suppose, however, that the agents operate under the following rule: Fix some small $\gamma > 0$. Let the sequence of communications and updates of tentative decisions take place until $|u^i_n - u^j_n| < \gamma$, $\forall i, j$ (small disagreement) and $|u^i_{n+1} - u^i_n| < \gamma$, $\forall i$ (small foreseeable changes in tentative decisions).

Then, we obtain:

Corollary 1: With the above rule and the assumptions of Theorems 1 or 2, the process terminates in finite time, with probability 1, for any $\gamma > 0$.

When $\Omega$ and $U$ are finite, we have:

Theorem 3: If $\Omega$ and $U$ are finite sets, if each agent communicates all the values of $v$ that minimize $E[c(v)|\mathcal{F}^i_n]$ and if assumption 6 holds, then there exists some positive integer $Q$ such that

$$u^i_Q = u^j_Q, \quad \forall i, j \quad \text{and} \quad u^i_{Q+n} = u^i_Q, \quad \forall i, \forall n, \forall \omega \in \Omega.$$

Strictly speaking, tentative decisions in the above theorem are not elements of $U$ but subsets of $U$. This is due to the non-uniqueness of optimal tentative decisions, in the absence of convexity. The equalities appearing in Theorem 3 have to be interpreted, therefore, as equalities of sets.

We now assume that the agents have perfect memory. We obtain results similar to theorems 1 and 2 under *much more relaxed* assumptions on the communications process. Namely, we only need to assume the following:

Assumption 9: Let $M^{ij}_k$ be the $k$-th message sent by agent $j$ to agent $i$. We assume that when agent $i$ receives $M^{ij}_k$, he knows that this is indeed the $k$-th message sent to him by agent $j$.

Remark: This assumption is trivially satisfied if messages arrive at exactly the same order as they are sent, with probability 1.

Theorem 4: Under assumptions 2.2 (convex costs), 8 (perfect memory) and 9, there exists a $U$-valued random variable $u^*$ such that $\lim_{n\to\infty} u^i_n = u^*$, $\forall i$, in probability and in $L_2(\Omega, \mathcal{F}, \mu)$.

For estimation problems ($u^i_n = E[x|\mathcal{F}^i_n]$), theorem 4 can be slightly strengthened: [5, Theorem 1]

Theorem 5: For estimation problems, under the assumptions of theorem 4, convergence to $u^*$ takes place with probability 1.

We now consider the case where $U$ is finite but (unlike theorem 3) $\Omega$ is allowed to be infinite. Several complications may arise, all of them due to the fact that optimal decisions, given some

information, are not guaranteed to be unique. We discuss these issues briefly, in order to motivate the next theorem.

Suppose that $U = \{v_1, v_2\}$. It is conceivable that $E[c(v_1)|\mathcal{F}_n^i] - E[c(v_2)|\mathcal{F}_n^i]$ is never zero and changes sign an infinite number of times, on a set of positive probability. In that case, the decisions of agent $i$ do not convege. Even worse, it is conceivable that $E[c(v_1)|\mathcal{F}_n^i] > E[c(v_2)|\mathcal{F}_n^i]$ and $E[c(v_1)|\mathcal{F}_n^j] < E[c(v_2)|\mathcal{F}_n^j]$, for all $n$ and for all $\omega$ in a set of positive probability, in which case agents $i$ and $j$ disagree foever. It is not hard to show that in both of the above cases $E[c(v_1)|\mathcal{F}_\infty^i] = E[c(v_2)|\mathcal{F}_\infty^i]$, on a set of positive probability and this non–uniqueness is the source of the pathology. The following theorem states that convergence and agreement are still obtained, provided that we explicitly exclude the possibility of non–uniqueness.

Theorem 6: Under assumptions 2.1 (finite $U$) and 8 (perfect memory) and if the random variable $u^i$ that minimizes $E[c(w)]$ over all $\mathcal{F}_\infty^i$-measurable random variables is unique up to a set of measure zero, for all $i$, then $\lim_{n\to\infty} u_n^i = u^i$, almost surely, and $u^i = u^j$, $\forall i, j$.

Although the preceding theorems guarantee that (under certain conditions) all agents will agree, nothing has been said concerning the particular decision to which all agents' decisions converge. In particular, it is not necessarily true, as one *would be tempted to conjecture*, that the limit decision is the optimal centralized solution (that is, the solution to be obtained if all agents were to communicate all their information). On the other hand, the centralized solution is reached for LQG problems, under the perfect memory assumption [5] and is also reached generically for an estimation problem on a finite probability space [6]. This issue will be touched again in the next section.

## IV. THE LINEAR QUADRATIC GAUSSIAN (LQG) MODEL.

In this section we specialize and strengthen some of our results by restricting to the Linear Quadratic Gaussian model described in section 2. (Recall that any such problem is equivalent to an estimation problem; therefore, $u_n^i = \hat{x}_i^n = E[x|\mathcal{F}_n^i]$, for some random vector $x$.) Theorems 1, 3 and 4 are applicable. Moreover, the results of [5] guarantee that, under assumption 8 (perfect memory), $u_n^i$ converges to the optimal centralized estimate, given the information possessed by all agents. The following theorem states that the same is true under the weaker assumption 7.

Theorem 7: For the LQG problem and under assumption 7 (imperfect memory; own data remembered) $\lim_{n\to\infty} \hat{x}_n^i = \hat{x}$, in the mean square, where $\hat{x} = E[x|\mathcal{F}_\infty]$ and $\mathcal{F}_\infty$ is the smallest $\sigma$-field containing $\mathcal{F}_n^i$, for all $i$, $n$.

Note that theorem 7 is much stronger than theorem 1 which was proved for the general case of imperfect memory. We have here convergence to a limit solution which is also guaranteed to be the optimal centralized solution.

Our final result concerns the finite dimensional LQG problem in which the total number of observations is finite. Namely, the smallest $\sigma$-field containing $\mathcal{F}_n^i$ for all $i$, $n$ is generated by a finite number of (jointly Gaussian) random variables. In that case, the centralized solution is going to be reached by all agents in a finite number of stages, provided that all agents have perfect memory.

Theorem 8: For the LQG problem with finitely many observations and under assumption 8 (perfect memory), the centralized solution is reached by all agents in a finite number of stages.

Theorems 7 and 8 imply that the scheme considered in this paper may be viewed as a distributed way of solving static linear estimation problems.

## V. CONCLUSIONS.

A set of agents with the same objective who start communicating to each other their tentative optimal decisions are guaranteed to agree in the limit. This is true even if their memory is limited and they are allowed to forget some of their past knowledge. Moreover, they are guaranteed to converge to the optimal centralized decision for linear estimation problems, provided that they do not forget their own observations. These results are valid when all agents share the same model of the world. The characterization of the behavior of agents with different models (perceptions) is an open problem.

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## VII. APPENDIX.

Proof of Lemma 1: By theorem 1.1.6 and corollary 1.1.7 of [9], there exists a condtional probability distribution $\{Q_\omega : \omega \in \Omega\}$ of $\mathcal{P}$ given $\mathcal{F}_n^i$ and, for any $v \in U$, a version of $E[c(v)|\mathcal{F}_n^i]$ such that

$$E[c(v)|\mathcal{F}_n^i] = \int_\Omega c(v)\, dQ_\omega. \tag{4}$$

Equation (4) and assumption 3 imply that $E[c(v)|\mathcal{F}_n^i]$ is a strictly convex function of $v$, diverging to infinity as $||v|| \to \infty$, $\forall \omega \in \Omega$.

By assumption 4, there exists a Polish probability space $(\Omega^*, \mathcal{F}^*, \mathcal{P}^*)$ and a mapping $f:\Omega \mapsto \Omega^*$ such that $\mathcal{F}_n^i$ is the smallest $\sigma$-field in $\mathcal{F}$ with respect to which $f$ is measurable. Moreover, $\mathcal{P}^*$ is the measure induced by $\mathcal{P}$; namely, $\mathcal{P}^*(B) = \mathcal{P}(f^{-1}(B))$, $\forall B \in \mathcal{F}^*$. We then use theorem T18 (p. 10) of [8] and the fact that $(\Omega^*, \mathcal{F}^*)$ is isomorphic to $(S, \mathcal{B})$, where $S \subset R$ and $\mathcal{B}$ is the Borel $\sigma$-field on $S$ [4, p.122], to conclude that there exists a measurable function $c^*: U \times \Omega^* \mapsto R$ such that

$$c^*(v, f(\omega)) = E[c(v)|\mathcal{F}_n^i](\omega) \tag{5}$$

Moreover, $c^*$ is strictly convex in $v$ and its minimum is attained for any fixed $\omega^* \in \Omega^*$. By the results of [4] on measurable selections (see in particular the discussion in page 259) there exists a universally measurable function $\hat{u}^*:\Omega^* \mapsto U$ such that

$$c^*(\hat{u}^*(\omega^*), \omega^*) \leq c^*(v, \omega^*), \qquad \forall v \in u,\ \forall \omega^* \in \Omega^*. \tag{6}$$

We now modify $\hat{u}^*(\omega^*)$ on a set of $\mathcal{P}^*$-measure zero, to obtain a Borel measurable function $u^*:\Omega^* \mapsto U$ such that

$$c^*(u^*(\omega^*), \omega^*) \leq c^*(v, \omega^*), \qquad \forall v \in U,\ \mathcal{P}^*\text{–almost surely} \tag{7}$$

We finally define $\hat{u}_n^i(\omega) = \hat{u}^*(f(\omega))$ and $u_n^i(\omega) = u^*(f(\omega))$. (Note that $u_n^i$ is $\mathcal{F}_n^i$-measurable, by construction, and that $\hat{u}_n^i(\omega)$ minimizes $E[c(v)|\mathcal{F}_n^i](\omega)$.) Let $B = \{\omega: u_n^i(\omega) \neq \hat{u}_n^i(\omega)\}$ and $D = \{\omega^*: u^*(\omega) \neq \hat{u}^*(\omega)\}$. Clearly, $B = f^{-1}(D)$ and $\mathcal{P}^*(D) = 0$. So, $0 = \mathcal{P}^*(D) = \mathcal{P}(f^{-1}(D)) = \mathcal{P}(B)$. Therefore, $u_n^i$ is an $\mathcal{F}_n^i$-measurable function that minimizes $E[c(v)|\mathcal{F}_n^i]$, almost everywhere, and this implies the first part of the Lemma.

For any $\mathcal{F}_n^i$-measurable, $U$-valued random variable $w$, $E[c(u_n^i)|\mathcal{F}_n^i] \leq E[c(w)|\mathcal{F}_n^i]$, almost surely, and by integrating, $E[c(u_n^i)] \leq E[c(w)]$.

Now suppose that $E[c(u_n^i)] = E[c(w)]$, for some $U$-valued, $\mathcal{F}_n^i$-measurable random variable $w$. Using the minimizing property of $u_n^i$, we have

$$E\left[c\left(\frac{u_n^i + w}{2}\right)\right] \geq \frac{1}{2}(E[c(u_n^i)] + E[c(w)]) \tag{8}$$

The converse inequality also holds, by the convexity of $c$. Therefore, (8) is actually an equality. Recalling the fact that $c$ is strictly convex, for all $\omega \in \Omega$, we conclude that $u_n^i = w$, almost surely.

The validity of the same results (except for uniqueness) under assumption 2.1 is immediate from the fact that the minimum of the finitely many random variables $\{E[c(v)|\mathcal{F}_n^i], v \in U\}$ is itself a random variable and a measurable selection of $u_n^i$ is straightforward. ∎

Lemma 2: Let $\{u_n\}$, $\{w_n\}$ be two sequences of $U$-valued random variables such that

$$\lim_{n\to\infty} E\left[c\left(\frac{u_n + w_n}{2}\right)\right] = \lim_{n\to\infty} E[c(u_n)] = \lim_{n\to\infty} E[c(w_n)] \tag{9}$$

If assumptions 2.2 and 3 hold, then $\lim_{n\to\infty}(u_n - w_n) = 0$ in $L_2(\Omega, \mathcal{F}, \mu)$ and in probability.

Proof: By assumptions 2.2, 3 and equation (9),

$$\lim_{n\to\infty} E[A(\omega)||u_n - w_n||^2] \leq \lim_{n\to\infty} E\left[\frac{c(u_n) + c(w_n)}{2} - c\left(\frac{u_n + w_n}{2}\right)\right] = 0 \tag{10}$$

which shows that $(u_n - w_n)$ converges to zero in $L_2(\Omega, \mathcal{F}, \mu)$. Therefore, it also converges in measure with respect to $\mu$.

Recall that $A(\omega) > 0$ and $\mu(B) = \int_B A(\omega)\, d\mathcal{P}(\omega)$, $\forall B \in \mathcal{F}$. Therefore, $\mu(B) = 0$ implies $\mathcal{P}(B) = 0$ and $\mathcal{P}$ is absolutely continuous with respect to $\mu$. Let $B_n^\epsilon = \{|u_n - w_n| \geq \epsilon\}$. Since $(u_n - w_n)$ converges to zero in measure $\mu$, for any $\epsilon > 0$, we have $\lim_{n\to\infty} \mu(B_n^\epsilon) = 0$ and, by absolute continuity, $\lim_{n\to\infty} \mathcal{P}(B_n^\epsilon = 0$, which shows that we have convergence in probability. ∎

Proof of Theorem 1: Since $u_n^i$ is $\mathcal{F}_n^i$-measurable, we have (by the minimizing property of $u_{n+1}^i$) $E[c(u_{n+1}^i)] \leq E[c(u_n^i)]$. Since $c$ is nonnegative, $E[c(u_n^i)]$ converges to some constant $g^i$. We also note that $(u_{n+1}^i + u_n^i)/2$ is $\mathcal{F}_{n+1}^i$-measurable and by taking the limit in the relation

$$\frac{1}{2}E[c(u_{n+1}^i) + c(u_n^i)] \geq E\left[c\left(\frac{u_{n+1}^i + u_n^i}{2}\right)\right] \geq E[c(u_{n+1}^i)] \tag{11}$$

we obtain $\lim_{n\to\infty} E[c((u_{n+1}^i + u_n^i)/2)] = g^i$. Lemma 2 then yields the first part of the theorem.

Let $j \in A(i)$. Then there exist sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $\lim_{k\to\infty} m_k = \lim_{k\to\infty} n_k = \infty$ and $m_k, n_k$ are the times of transmission and reception, respectively, of the $k$-th message from agent $j$ to agent $i$. Therefore, $u_{m_k}^j$ is $\mathcal{F}_{n_k}^i$-measurable, for all $k$, and $E[c(u_{n_k}^i)] \leq E[c(u_{m_k}^j)]$ which shows that $g^j \leq g^i$. Using assumption 5, we conclude that $g^i = g^j$, $\forall i, j$.

We note that $(u_{n_k} + u_{m_k}^j)/2$ is $\mathcal{F}_{n_k}^i$-measurable and, therefore,

$$\frac{1}{2}E[c(u_{n_k}^i) + c(u_{m_k}^j)] \geq E\left[c\left(\frac{u_{n_k}^i + u_{m_k}^j}{2}\right)\right] \geq E[c(u_{n_k}^i)] \tag{12}$$

Taking the limit in (12) and using Lemma 2, we obtain the second half of the theorem. ∎

Proof of Theorem 2: Theorem 1 and the discussion preceding the statement of theorem 2 show that $\lim_{n\to\infty}(u_{n+1}^i - u_n^i) = 0$, in probability with respect to $\mathcal{P}$, for all $\omega^* \in \Omega^*$. Let $\chi_n(\omega, \omega^*)$ be the characteristic function of the set $\{(\omega, \omega^*): \|u_{n+1}^i - u_n^i\| < \epsilon\}$. Then,

$$\lim_{n\to\infty} \int \chi_n d(\mathcal{P} \times \mathcal{P}^*) = \lim_{n\to\infty} \int\int \chi_n d\mathcal{P}\, d\mathcal{P}^* = \int \lim_{n\to\infty} \int \chi_n d\mathcal{P}\, d\mathcal{P}^* = 1 \tag{13}$$

(The first equality holds by the Fubini theorem; the second by the dominated convergence theorem; the third by convergence of $(u_{n+1}^i - u_n^i)$ to zero, with respect to the probability measure $\mathcal{P}$.) This shows that $u_{n+1}^i - u_n^i$ converges to zero in probability with respect to $\mathcal{P} \times \mathcal{P}^*$. ∎

Proof of Corollary 1: By theorem 1, the $U \times U$-valued sequence of random variables $(u_n^i - u_n^j, u_{n+1}^i - u_n^i)$ converges to $(0,0)$ in probability. It therefore contains a subsequence converging to $(0,0)$, almost surely [10]. Therefore, $\forall \gamma > 0$ and for almost all $\omega \in \Omega$, $\exists n_o$ such that $|u_{n_o}^i - u_{n_o}^j| < \gamma$, $|u_{n_o+1}^i - u_{n_o}^i| < \gamma$ and the termination condition is eventually satisfied with probability one. ∎

Proof of Theorem 3: Because of the finiteness of $\Omega$, there exists a finite (non-random) time after which communications (conditioned on past events) are deterministic. We may take that time as the initial time and assume, without loss of generality, that all communications are deterministic.

Let $u_n^i$ be the set of elements of $U$ which are optimal, given $\mathcal{F}_n^i$. Let $w_n^i$ denote $\mathcal{F}_n^i$-measurable random variable such that $w_n^i(\omega) \in u_n^i(\omega)$, $\forall \omega \in \Omega$. (Note that $E[c(w_n^i)]$ is independent of how $w_n^i$ has been selected.) By finiteness of $\Omega$ and $U$, there exist finitely many $U$-valued random variables and, *since* $E[c(w_{n+1}^i)] \leq E[c(w_n^i)]$, we conclude that there exists some positive integer $T$ and

some $g^i$ such that $E[c(w_n^i)] = g^i$, $\forall n > T$. For any $n > T$, $E[c(w_n^i)] = E[c(w_{n+1}^i)]$ and since $w_n^i$ is $\mathcal{F}_{n+1}^i$-measurable, $w_n^i$ minimizes $E[c(w)]$ over all $\mathcal{F}_{n+1}^i$-measurable random variables. Hence, $w_n^i(\omega) \in u_{n+1}^i(\omega)$, $\forall \omega \in \Omega$ which shows that $u_n^i(\omega) \subset u_{n+1}^i(\omega)$, $\forall \omega \in \Omega$. Again, by finiteness of $U$ and $\Omega$, there exists some positive integer $Q$ such that $u_{n+1}^i(\omega) = u_n^i(\omega)$, $\forall n > Q$, $\forall \omega \in \Omega$, $\forall i$.

If $j \in A(i)$, there exist $m, n > Q$ such that $w_m^j$ is $\mathcal{F}_n^i$-measurable and this shows that $g^j \leq g^i$. By assumption 5, we obtain $g^i = g^j$, $\forall i, j$. Therefore, $w_m^j$ minimizes $E[c(w)]$ over all $\mathcal{F}_n^i$-measurable random variables and, therefore, $w_m^j(\omega) \in u_n^i(\omega)$, or, $u_m^j(\omega) \subset u_n^i(\omega)$, $\forall \omega \in \Omega$. Recalling assumption 5, we obtain $u_m^j(\omega) = u_n^i(\omega)$, $\forall i, j$, $\forall m, n > Q$, $\forall \omega$. ∎

Lemma 3: Let $T$ be a finite stopping time of an increasing family $\{\mathcal{F}_n\}$ of $\sigma$-fields. Let $u_n$, $n = 1, 2, \ldots$ be random variables that minimize $E[c(w)|\mathcal{F}_n]$, almost surely, over all $\mathcal{F}_n$-measurable random variables $w$. Then, $u_T$ minimzes $E[c(w)]$ over all $\mathcal{F}_T$-measurable random variables $w$, where $u_T = u_n$ if and only if $T = n$.

Proof: Let $\chi_n$ be the indicator function of the set $\{\omega : T(\omega) = n\}$. Since $T$ is a stopping time, $\chi_n$ is $\mathcal{F}_n$-measurable. Note that $\chi_n c(u_n) = \chi_n c(u_T)$. Let $w$ be a $\mathcal{F}_T$-measurable random variable and note that $\chi_n c(w) = \chi_n c(\chi_n w)$ and $\chi_n w$ is $\mathcal{F}_n$-measurable. Therefore,

$$E[\chi_n c(w)|\mathcal{F}_n] = \chi_n E[c(\chi_n w)|\mathcal{F}_n] \geq \chi_n E[c(u_n)|\mathcal{F}_n] = E[\chi_n c(u_n)|\mathcal{F}_n] = E[\chi_n c(u_T)|\mathcal{F}_n] \tag{14}$$

Taking expectations, we obtain

$$E[\chi_n c(w)] \geq E[\chi_n c(u_T)] \tag{15}$$

and summing over all $n$'s (and using the monotone convergence theorem to interchange summation and expectation) we obtain $E[c(w)] \geq E[c(u_T)]$. ∎

Proof of Theorem 4: Since $u_n^i$ is $\mathcal{F}_{n+1}^i$-measurable, we have $E[c(u_{n+1}^i)] \leq E[c(u_n^i)]$. Since $c$ is nonnegative, $E[c(u_n^i)]$ converges to some constant $g^i$. We also note that $(u_n^i + u_{n+m}^i)/2$ is $\mathcal{F}_{n+m}^i$-measurable. Therefore, $E[c((u_{n+m}^i + u_n^i)/2)] \geq E[c(u_{n+m}^i)] \geq g^i$. Fix some $\epsilon > 0$, and let $n$ be large enough so that $E[c(u_n^i)] \leq g^i + \epsilon$. Then, using assumption 3, we obtain $E[A(\omega)||u_{n+m}^i - u_n^i||^2] \leq \epsilon$, $\forall m \geq 0$. Therefore, $\{u_n^i\}$ is a Cauchy sequence in $L_2(\Omega, \mathcal{F}, \mu)$. By the closedness of $U$ and completeness of $L_2$ spaces, there exists a $U$-valued random variable $u^i$ such that $\lim_{n\to\infty} u_n^i = u^i$, in $L_2(\Omega, \mathcal{F}, \mu)$ and, therefore, in probability, with respect to $\mathcal{P}$. (The proof of the last implication is contained in the proof of Lemma 2.)

Fix some $v \in U$. Then, $\{E[c(v)|\mathcal{F}_n^i],\ n = 1, 2, \ldots\}$ is a uniformly integrable martingale [8, Theorem T19, p.86]. Since $E[c(u_n^i)|\mathcal{F}_n^i] \leq E[c(v)|\mathcal{F}_n^i]$, a.s., and

$$E[E[c(u_{n+1}^i)|\mathcal{F}_{n+1}^i]|\mathcal{F}_n^i] \leq E[E[c(u_n^i)|\mathcal{F}_{n+1}^i]|\mathcal{F}_n^i] = E[c(u_n^i)|\mathcal{F}_n^i], \tag{16}$$

$\{E[c(u_n^i)|\mathcal{F}_n^i],\ n = 1, 2, , \ldots\}$ is a uniformly integrable supermartingale, with respect to $\{\mathcal{F}_n^i\}$.

Let $j \in A(i)$. Let $m_k$, $n_k$ be the times of transmission and reception, respectively, of the $k$-th message from $j$ to $i$. Because of assumption 9, both $m_k$ and $n_k$ are stopping times of $\{\mathcal{F}_n^j\}$, $\{\mathcal{F}_n^i\}$, respectively, and since $j \in A(i)$, they are almost surely finite stopping times, for all $k$. Moreover, $k \leq m_k \leq n_k$ and by the optional sampling theorem [8, Theorem T28, p.90]

$$E[c(u_k^i)] \geq E[c(u_{n_k}^i)] \geq g^i$$

which shows that $\lim_{k\to\infty} E[c(u_{n_k}^i)] = g^i$. Similarly, $\lim_{k\to\infty} E[c(u_{m_k}^j)] = g^j$.

Note that $u_{m_k}^j$ is $\mathcal{F}_{n_k}^i$-measurable and, by Lemma 3, $E[c(u_{m_k}^j)] \geq E[c(u_{n_k}^i)]$. Taking the limit, we obtain $g^j \geq g^i$, and by assumption 5, $g^i = g^i = g$, $\forall i, j$.

We now take the limit of the inequalities

$$\frac{1}{2} E[c(u_{n_k}^i) + c(u_{m_k}^j)] \geq E\left[c\left(\frac{u_{n_k}^i + u_{m_k}^j}{2}\right)\right] \geq E[c(u_{n_k}^i)] \tag{17}$$

to obtain $\lim_{k\to\infty} E[c((u_{n_k}^i + u_{m_k}^j)/2)] = g$ and, by Lemma 2, $\lim_{k\to\infty}(u_{n_k}^i - u_{m_k}^j) = 0$, in $L_2(\Omega, \mathcal{F}, \mu)$ and in probability.

We also take the limit of the inequalities

$$\frac{1}{2} E[c(u_{n_k}^i) + c(u_k^i)] \geq E\left[c\left(\frac{u_{n_k}^i + u_k^i}{2}\right)\right] \geq E[c(u_{n_k}^i)] \tag{18}$$

to obtain $\lim_{k\to\infty} E[c((u_k^i + u_{n_k}^i)/2)] = g^i$ and, by Lemma 2, $\lim_{k\to\infty}(u_{n_k}^i - u_k^i) = 0$. Similarly, we obtain $\lim_{k\to\infty}(u_{m_k}^j - u_k^j) = 0$, which shows that $u^i = u^j$, almost surely. ∎

Proof of Theorem 6: Fix some $v \in U$ and let $B = \{\omega : u^i(\omega) = v\}$. Then, $E[c(v)|\mathcal{F}_\infty^i] < E[c(v^*)|\mathcal{F}_\infty^i]$, $\forall v^* \in U$, for almost all $\omega \in B$. By the martingale convergence theorem [8, Theorem T17, p.84], we conclude that, for almost all $\omega \in B$, there exists some $N(\omega)$ such that

$$E[c(v)|\mathcal{F}_n^i] < E[c(v^*)|\mathcal{F}_n^i] \qquad \forall n \geq N(\omega)$$

Therefore, $u_n^i(\omega) = v$, for almost all $\omega \in B$ and, by considering the other elements of $U$ as well, $\lim_{n\to\infty} u_n^i = u^i$, almost surely.

If $j \in A(i)$, $u^j$ is $\mathcal{F}_\infty^i$-measurable and $E[c(u^j)] \geq E[c(u^i)]$. By assumption 5, $E[c(u^i)] = E[c(u^j)]$, $\forall i, j$. Therefore, for $j \in A(i)$, $u^j$ minimizes $E[c(w)]$ over all $\mathcal{F}_\infty^i$-measurable random variables and by the assumptions of the theorem, $u^i = u^j$, almost surely. Using assumption 5 once more, we obtain $u^i = u^j$, $\forall i, j$. ∎

Proof of Theorem 7: As is usual in linear least squares estimation, we use the setting of Hilbert spaces of square integrable random variables. Let $G$ be a Hilbert space of zero mean, jointly Gaussian random variables on $(\Omega, \mathcal{F}, \mathcal{P})$ such that each component of the unknown vector $x$ and the observations belong to $G$. The inner product in $G$ is defined by $< x, y >= E[xy]$.

For each agent $i$, let $H^i$ denote the smallest closed subspace of $G$ containing all observations obtained by him. Let $H_n^i$ be the smallest closed subspace of $G$ containing all observations obtained by agent $i$ up to time $n$. (Note that $H_n^i$ does not contain all random variables known by agent $i$ at time $n$, because it does not need to contain any of the messages received by agent $i$. Note also that, by assumption 7, $H_n^i \subset H_{n+1}^i \subset H^i$ and that $\sum_{k=1}^N H_n^k$ is the total knowledge available to all agents, where $\sum$ denotes the direct sum. The centralized estimate is the projection of $x$ on $\sum_{k=1}^N H^k$.) We assume, without loss of generality, that $x$ is a scalar random variable, since each component can be separately estimated.

Let $\hat{x}_n^i = E[x|\mathcal{F}_n^i]$ and $e_n^i = x - \hat{x}_n^i$ and, by the orthogonality of errors and observations, we have $E[xy] = E[\hat{x}_n^i y]$, $\forall y \in H_n^i$. Similarly with previous proofs, we have $||e_{n+1}^i||^2 \leq ||e_n^i||^2$, $\forall n, i$ which implies that

$$||x||^2 \geq ||\hat{x}_{n+1}^i||^2 \geq ||\hat{x}_n^i||^2 \tag{19}$$

In particular, (19) implies that $\{\hat{x}_n^i\}$ is a norm-bounded sequence. By the weak local sequential compactness of Hilbert spaces [11, p.126], $\{\hat{x}_n^i\}$ contains a weakly convergent subsequence $\{\hat{x}_{n_l}^i\}$. In other words, there exists an element $\hat{x}_\infty \in G$ such that $< y, \hat{x}_{n_l}^i >$ converges to $< y, \hat{x}_\infty^i >$, $\forall y \in G$. Moreover, $\hat{x}_n^i \in \sum_{k=1}^N H_n^k \subset \sum_{k=1}^N H^k$ and since closed subspaces are also weakly closed [11, Theorem 11, p.125], $\hat{x}_\infty \in \sum_{k=1}^N H^k$. Now let $y \in H_n^i$. Then, $< y, \hat{x}_{n_l}^i >=< y, x >$, for all $l$ such that $n_l \geq n$, which implies that $< y, \hat{x}_\infty >=< y, x >$. Moreoever, the sequence of spaces $\{H_n^i\}$ generates $H^i$ which implies that $< y, \hat{x}_\infty >=< y, x >$, $\forall y \in H^i$.

By theorem 1, $(x_n^i - x_n^j)$ converges in mean square (and therefore weakly) to zero, which implies that $\hat{x}_\infty$ is also a weak limit point of $\{\hat{x}_{n_l}^j\}$. The same argument as before shows that $< y, \hat{x}_\infty >=$ $< y, x >$, $\forall y \in H^j$, $\forall j$. Therefore, $< y, \hat{x}_\infty >= < y, x >$, $\forall y \in \sum_{k=1}^N H^k$. But this is exactly the

condition that $\hat{x}_\infty$ is the centralized estimate, given the observations of all agents. So, $\{\hat{x}_n^i\}$ has a unique weak limit point, which is the same for all $i$ and coincides with the centralized estimate.

It only remains to show that $\hat{x}_n^i$ converges to $\hat{x}_\infty$ strongly (in the mean square). We know from [11, p.120] that $\|\hat{x}_\infty\| \leq \liminf_{n\to\infty} \|\hat{x}_n^i\|$. On the other hand,

$$\|x\|^2 - \|\hat{x}_\infty\|^2 = \|x - \hat{x}_\infty\|^2 \leq \|x - \hat{x}_n^i\|^2 = \|x\|^2 - \|\hat{x}_n^i\|^2 \tag{20}$$

which shows that $\|\hat{x}_\infty\| \geq \limsup_{n\to\infty} \|\hat{x}_n^i\|$. Therefore, $\|\hat{x}_\infty\| = \lim_{n\to\infty} \|\hat{x}_n^i\|$ and by theorem 8, p.124 of [11], we conclude that $\lim_{n\to\infty} \|\hat{x}_n^i - \hat{x}_\infty\|^2 = 0$. ∎

<u>Proof of Theorem 8:</u> We use again the Hilbert space formalism of the previous proof. Let $G_n^i$ be the subspace of $G$ describing the knowledge of agent $i$ at time $n$ (both his observations and the messages he has received). By assumption 8, we have $G_n^i \subset G_{n+1}^i \subset G$. Since $G$ can be chosen to be finite dimensional, there exists some $M$ (depending on the sequence of communications but deterministic) such that $G_{M+n}^i = H_M^i$, $\forall n \geq 0$, $\forall i$. Equivalently, $\hat{x}_{M+n}^i = \hat{x}_M^i$, $\forall n \geq 0$, $\forall i$, and by theorem 7, $\hat{x}_M^i = \hat{x}_M^j = \hat{x}_\infty$, $\forall i, j$. ∎

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